Corrigé Session 2017.

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Q1

$$f(x) = \frac{x}{\sqrt{1+x^2}} \qquad \mathcal{D}_f = \mathbb{R}$$

(A) VRAI

$$f'(x) = \frac{1 \cdot \sqrt{1+x^2} - x \frac{2x}{2\sqrt{1+x^2}}}{\left(\sqrt{1+x^2}\right)^2}$$

$$= \frac{1}{1+x^2} \times \frac{\sqrt{1+x^2}^2 - x^2}{\sqrt{1+x^2}}$$

$$= \frac{1+x^2-x^2}{(1+x^2)\sqrt{1+x^2}}$$

$$f'(x) = \frac{1}{(1+x^2)\sqrt{1+x^2}} \qquad \forall x \in \mathbb{R}$$

(B) VRAI

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{x}{\sqrt{x^2}} = \lim_{x \to +\infty} \frac{x}{x} = 1$$

(C) FAUX

$$\arcsin'(u) = \frac{u'}{\sqrt{1-u^2}}$$

(D) VRAI

$$g(x) = \arcsin(f(x))$$

$$g'(x) = f'(x) \times \frac{1}{\sqrt{1-f^2(x)}}$$

$$= \frac{1}{(1+x^2)\sqrt{1+x^2}} \times \frac{1}{\sqrt{1-\frac{x^2}{1+x^2}}}$$

$$= \frac{1}{(1+x^2)\sqrt{1+x^2}} \times \frac{1}{\sqrt{\frac{1+x^2-x^2}{1+x^2}}}$$

$$g'(x) = \frac{1}{(1+x^2)\sqrt{1+x^2}} \times \frac{\sqrt{1+x^2}}{\sqrt{1}} = \frac{1}{1+x^2}$$

donc $g'(x) = \arctan'(x) \quad \forall x \in \mathbb{R}$

(E) $g'(x) = \arctan'(x) \quad \forall x \in \mathbb{R}$
FAUX

$\Rightarrow g(x) = \arctan x + cte \quad \forall x \in \mathbb{R}$.

$g(1) = \arcsin(f(1)) = \arcsin\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

et $\arctan(1) + cte = \frac{\pi}{4} + cte$

Donc $\frac{\pi}{4} = \frac{\pi}{4} + cte \Leftrightarrow cte = 0$

et $g(x) = \arctan x \quad \forall x \in \mathbb{R}$

Q2

(A) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + x^3 \varepsilon(x)$ avec $\varepsilon(x) \xrightarrow[x \to 0]{} 0$
VRAI

$\Rightarrow e^x - 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + x^3 \varepsilon(x)$.

(B) $f(x) = \begin{cases} \frac{x}{e^x - 1} & \text{si } x \neq 0 \\ 1 & \text{si } x = 0 \end{cases}$
VRAI

$\lim_{x \to 0} f(x) = \lim_{x \to +\infty} \frac{x}{x} = 1 = f(0)$
d'après (A)

donc f est continue en 0.

(C) $\frac{1}{1+u} = 1 - u + u^2 + u^2 \varepsilon(u)$
FAUX

(D) $f(x) = \frac{x}{\sqrt{1+x^2}}$
FAUX

$\sqrt{1+u} = 1 + \frac{1}{2}u - \frac{1}{8}u^2 + u^2 \varepsilon(u)$
donc

$\sqrt{1+x^2} = 1 + \frac{1}{2}x^2 + x^2 \varepsilon(x)$ (à l'ordre 2)

Alors $\frac{1}{\sqrt{1+x^2}} = \frac{1}{1 + \underbrace{\frac{1}{2}x^2 + x^2 \varepsilon(x)}_{u}} = 1 - \frac{1}{2}x^2 + x^2 \varepsilon(x)$ à l'ordre 2

et

$\frac{x}{\sqrt{1+x^2}} = x - \cancel{\frac{1}{2}x^3 + x^3 \varepsilon(x)}$ DL en 0

à l'ordre 2 on obtient : $\frac{x}{\sqrt{1+x^2}} = x + x^2 \varepsilon(x)$

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(E) $f(x) = x + x^2 \varepsilon(x)$

VRAI

Donc $f(x) - \underset{\uparrow}{x} = x^2 \varepsilon(x) > 0 \quad \forall x$ proche de 0

$y = x$ est l'équat° de la tgte à C_f en 0

Ainsi C_f est au-dessus de sa tangente en 0, au voisinage de 0

Q3

(A) FAUX

$L = \int_0^1 \ln x \, dx = \lim_{X \to 0^+} \int_X^1 \ln x \, dx$

IPP: $u = \ln x \quad u' = \frac{1}{x}$

$v' = 1 \quad v = x$

donc $L = \lim_{X \to 0^+} \left\{ \left[x \ln x \right]_X^1 - \int_X^1 1 \, dx \right\}$

$= \lim_{X \to 0^+} \left(\ln 1 - X \ln X - \left[x \right]_X^1 \right)$

$= \lim_{X \to 0^-} \left(X \ln X - 1 + X \right)$

$L = -1$ donc L converge

(B) VRAI

$I = \int_0^{\pi/2} \ln(\tan x) \, dx$

$= \underbrace{\int_0^{\pi/2} \ln\left(\frac{\sin x}{\cos x}\right) dx}_{\ln(\sin x) - \ln(\cos x)} = J - K$

(C) VRAI

$t = \tan x$

$dt = (1 + \tan^2 x) \, dx$

$\frac{dt}{1+t^2} = dx$

$x = 0 \iff t = 0$

$x = X \Rightarrow t = \tan X$

$I = \lim_{\substack{X \to \pi/2 \\ X < \pi/2}} \int_0^X \ln(\tan x) \, dx$

$I = \lim_{\substack{X \to \frac{\pi}{2} \\ X < \frac{\pi}{2}}} \int_0^{\tan X} \frac{\ln(t)}{1+t^2} \, dt$

$J = \int_0^{+\infty} \frac{\ln(t)}{1+t^2} \, dt$

(D) FAUX $\left(\arctan(\ln t)\right)' = \frac{1}{t} \times \frac{1}{1+\ln^2 t}$

(E) FAUX
$$\int_0^1 \frac{\ln t}{1+t^2}\,dt = \lim_{x\to 0^+} \int_x^1 \frac{\ln t}{1+t^2}\,dt$$

$t = \frac{1}{u} \Rightarrow dt = -\frac{1}{u^2}\,du$

$$= \lim_{x\to 0^+} \int_{\frac{1}{x}}^1 \frac{\ln(1/u)}{1+(1/u)^2} \times \frac{-1}{u^2}\,du = \lim_{x\to 0^+} \int_{1/x}^1 \frac{\ln u}{u^2+1}\,du$$

$$= \int_{+\infty}^1 \frac{\ln u}{1+u^2}\,du$$

donc $\int_0^1 \frac{\ln t}{1+t^2}\,dt = -\int_1^{+\infty} \frac{\ln u}{1+u^2}\,du$

Q4) (A) VRAI $I = \int_0^\infty \frac{\ln t}{1+t^2}\,dt = \underbrace{\int_0^1 \frac{\ln t}{1+t^2}\,dt}_{} + \int_1^{+\infty} \frac{\ln t}{1+t^2}\,dt$

$$= -\int_1^{+\infty} \frac{\ln t}{1+t^2}\,dt$$

donc $I = 0$

(B) VRAI Comme $I = J - K = 0$ alors $J = K$

(C) FAUX $\underbrace{J+K}_{2J} = \int_0^{\pi/2} \left(\underbrace{\ln(\sin x) + \ln(\cos x)}\right) dx \neq \int_0^{\pi/2} \ln(\sin(2x))\,dx$

$\| \ \ln(\underbrace{\sin x \times \cos x})$

$\| \ \ln\left(\frac{1}{2}\sin(2x)\right) = \ln(\sin 2x) - \ln 2$

$2J = \int_0^{\pi/2} \ln(\sin(2x))\,dx - \int_0^{\pi/2} \ln 2\,dx$ .. On pose $t = 2x$, $dt = 2dx$

$2J = -\frac{\pi}{2}\ln 2 + \int_0^{\pi} \ln(\sin t)\,\frac{dt}{2} = \frac{1}{2}\int_0^{\pi} \ln(\sin t)\,dt - \frac{\pi}{2}\ln 2$

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(D) VRAI

$$\int_0^{\pi} \ln(\sin t)\,dt = \underbrace{\int_0^{\pi/2} \ln(\sin t)\,dt}_{J} + \underbrace{\int_{\pi/2}^{\pi} \ln(\sin t)\,dt}_{\substack{x = \pi - t \\ dx = -dt}}$$

$$\int_{\pi/2}^{0} -\ln\big(\underbrace{\sin(\pi - x)}_{\sin x}\big)\,dx$$

$$= \int_0^{\pi/2} \ln(\sin x)\,dx$$

$$= J$$

donc $\displaystyle\int_0^{\pi} \ln(\sin t)\,dt = 2J.$

(E) Donc VRAI

$$2J = -\frac{\pi}{2}\ln 2 + \underbrace{\frac{1}{2}\int_0^{\pi} \ln(\sin t)\,dt}$$ d'après (C)

et $= \frac{1}{2} \times 2J$ d'après (D)

donc $2J - J = -\frac{\pi}{2}\ln 2 \Longrightarrow J = K = -\frac{\pi}{2}\ln 2$

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Q5

(A) $z = x+iy$

FAUX $p(z) = (x+iy)^3 + 3i(x+iy)^2 + (4i-9)(x+iy) - 15i$

alors: $z^3 - 9z$ $\quad -9(x+iy)$

$= x^3 + 3iyx^2 + 3i^2y^2x + i^3y^3 = x^3 - 3y^2x + 3iyx^2 - 3iy^3$

$\hookrightarrow -9x - 9iy \notin \mathbb{R}$

(B) FAUX

soit $a \in \mathbb{R}$ $p(a) = a^3 + 3ia^2 + (4i-9)a - 15i$

$= a^3 - 9a + i(3a^2 + 4a - 15)$

$p(A) = 0 \iff \begin{cases} a^3 - 9a = 0 \iff a(a^2-9) = 0 \iff \begin{cases} a=0 \\ a=3 \\ a=-3 \end{cases} \\ I(a) = 3a^2 + 4a - 15 = 0 \end{cases}$

$I(0) \neq 0$; $I(3) = 27 + 12 - 15 \neq 0$

$I(-3) = 27 - 12 - 15 = 0$

Donc $a = -3$ est l'unique racine réelle

(C) $\Delta = [3(i-1)]^2 - 4(-5i) = 9(i^2 - 2i + 1) + 20i$

VRAI $= -9 - 18i + 9 + 20i$

$\Delta = 2i$

(D) $\Delta = 2e^{i\pi/2} = (\sqrt{2}e^{i\pi/4})^2$ donc les racines carrées de $2i$ sont $\pm\sqrt{2}e^{i\pi/4} = \pm(1+i)$

VRAI

(E) $z_1 = \dfrac{-3(i-1) - \sqrt{2}e^{i\pi/4}}{2} = \dfrac{-3i + 3 - (1+i)}{2} = \dfrac{-4i+2}{2} = 1 - 2i$

VRAI

$z_2 = \dfrac{-3(i-1) + \sqrt{2}e^{i\pi/4}}{2} = \dfrac{-3i + 3 + 1 + i}{2} = \dfrac{4 - 2i}{2} = 2 - i$

Q6

$3y'(x) + xy(x) = 0 \quad (H)$

$3y'(x) + xy(x) = x^3 \quad (S)$

(A) VRAI $3y'(x) + xy(x) = 0$

$$3\frac{dy}{dx} = -xy$$

$$\frac{dy}{y} = -\frac{x}{3}dx \iff \int \frac{dy}{y} = -\frac{1}{3}\int x\,dx$$

$$\Rightarrow \ln|y| = -\frac{1}{3}\frac{x^2}{2} + C \Rightarrow |y(x)| = \underbrace{e^{c}}_{\text{cte}>0} \times e^{-x^2/6}$$

$|K|$

$$\iff y(x) = Ke^{-x^2/6}$$

et comme $f(x) = K.e^{-x^2/6}$ avec $f(0) = 1$, alors

$$f(x) = e^{-x^2/6}$$

(B) FAUX $y_0(x) = A_0(x).f(x)$

$$y_0'(x) = A_0'(x)f(x) + A_0(x)f'(x)$$

$$y_0' = A_0'e^{-x^2/6} + A_0\left(-\frac{2x}{6}e^{-x^2/6}\right)$$

On remplace dans (S):

$$3A_0'e^{-x^2/6} \cancel{- A_0 x e^{-x^2/6}} \cancel{+ x A_0 e^{-x^2/6}} = x^3$$

$$3A_0'e^{-x^2/6} = x^3 \iff A_0' = \frac{1}{3}x^3 e^{x^2/6}$$

(C) VRAI

$$A_0(x) = \int_0^x \frac{1}{3}t^3 e^{t^2/6}\,dt = 6 \times \int_0^x t.\frac{t^2}{6}e^{t^2/6}\frac{dt}{3}$$

$$u = t^2/6 \iff du = \frac{t}{3}dt \qquad \begin{cases} t = 0 \iff u = 0 \\ t = x \iff u = x^2/6 \end{cases}$$

$$A_0(x) = \int_0^{x^2/6} u e^u\,du$$

(D) VRAI

IPP : $U = u$ $U' = 1$

$V' = e^u$ $V = e^u$

$$A_0(x) = \left[ue^u\right]_0^{x^2/6} - \int_0^{x^2/6} e^u\,du = \frac{x^2}{6}e^{x^2/6} - 0 - \left[e^u\right]_0^{x^2/6}$$

$$A_0(x) = \frac{x^2}{6}e^{x^2/6} - e^{x^2/6} + 1 = 1 + \left(\frac{x^2}{6} - 1\right)e^{x^2/6}$$

(E) Alors les soluts de (S) sont de la

FAUX forme : $y(x) = A_0(x) f(x)$

$$y(x) = \left(1 + \left(\frac{x^2}{6} - 1\right) e^{x^2/6}\right) K e^{-x^2/6}$$

$$y(x) = K e^{-x^2/6} + \frac{x^2}{6} - 1$$

et la solution vérifiant $y(0) = 0$ est telle que.

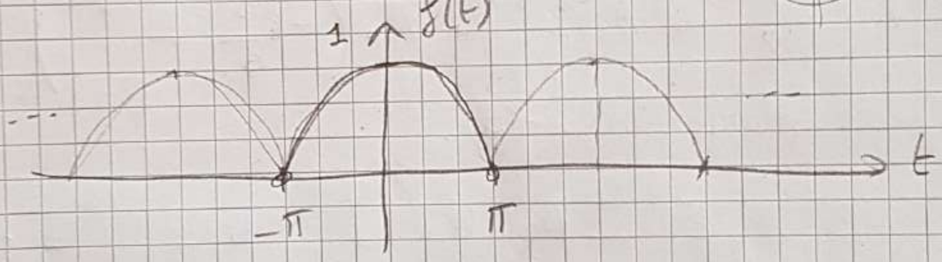
$$y(0) = K e^0 + 0 - 1 = 0 \iff K = 1$$

Donc
$$y_1(x) = e^{-x^2/6} + \frac{x^2}{6} - 1$$

Q11 $f(t) = \cos(t/2) \quad \forall t \in]-\pi;\pi]$

(A)

VRAI



$f \in C^0(\mathbb{R})$

(B) Dirichlet hyp1 $f \in C^0(]-\pi;\pi])$

VRAI

hyp2 f est dérivable sur $]-\pi;\pi]$, sauf en π où $f'(\pi^-) = -\frac{1}{2} \sin \pi/2 = -\frac{1}{2}$ est finie

$f'(t) = -\frac{1}{2} \sin(t/2)$ sur $]-\pi;\pi[$

et $f'(\pi^+) = \frac{1}{2}$ est finie

Donc S_f converge et a pour somme $S(t) = f(t) \ \forall t \in \mathbb{R}$

(C) $\cos(a+b) + \cos(a-b) = \cos a \cos b - \sin a \sin b + \cos a \cos b + \sin a \sin b$

VRAI $\cos(a+b) + \cos(a-b) = 2 \cos a \cos b$

(D) FAUX $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(t/2)\,dt =$

$$a_0 = \frac{2}{2\pi}\left[\sin(t/2)\right]_{-\pi}^{\pi} = \frac{2}{\pi}$$

FAUX (E) Comme f est paire, alors $b_n = 0 \ \forall n \in \mathbb{N}$

$\omega = \frac{2\pi}{T} = 1$

$$f(t) = a_0 + \sum_{k=1}^{+\infty} a_k \cos(kt)$$

$$a_k = \frac{2}{T}\int_{-\pi}^{\pi} f(t)\cos(kt)\,dt$$

$$= \frac{4}{\pi}\int_0^{\pi} \cos(t/2)\cos(kt)\,dt$$

$$= \frac{2}{2\pi}\int_0^{\pi}\left(\cos\left(\left(\tfrac{1}{2}+k\right)t\right) + \cos\left(\left(\tfrac{1}{2}-k\right)t\right)\right)dt$$

idem

$$= \frac{1}{\pi}\left[\frac{\sin((k+1/2)t)}{k+1/2} + \frac{\sin((k-1/2)t)}{k-1/2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\sin(k\pi+\pi/2)}{k+1/2} + \frac{\sin(k\pi-\pi/2)}{k-1/2}\right)$$

$$a_k = \frac{1}{\pi}\left(\frac{(-1)^k}{k+1/2} - \frac{(-1)^k}{k-1/2}\right)$$

$$a_1 = \frac{1}{\pi}\left(\frac{-1}{3/2} - \frac{-1}{1/2}\right) \neq 0.$$

Q12

(A) VRAI $a_k = \frac{1}{\pi}(-1)^k \frac{k-1/2-(k+1/2)}{k^2-1/4} \times \frac{4}{4}$

$$a_k = \frac{(-1)^k}{\pi}\,\frac{-4}{4k^2-1} \quad \forall k \neq 0$$

(B) VRAI $S(x) = \frac{2}{\pi} - \frac{4}{\pi}\cdot\sum_{k=1}^{\infty}\frac{(-1)^k}{4k^2-1}\cos(kx) = f(x) \ \forall x \in \mathbb{R}$

En particulier pour $x=0$: $\frac{2}{\pi} - \frac{4}{\pi}\cdot\sum_{k\geq 1}\frac{(-1)^k}{4k^2-1} = f(0) = 1$

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Donc $\sum_{k\geq 1} \frac{(-1)^k}{4k^2-1} = \left(1-\frac{2}{\pi}\right) \times -\frac{\pi}{4}$

$= \frac{1}{2} - \frac{\pi}{4}$

(C) $S(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{4k^2-1} \cos(kx) = f(x) \quad \forall x \in \mathbb{R}$

FAUX

En particulier pour $x=\pi$:

$$\frac{2}{\pi} - \frac{4}{\pi} \sum_{k\geq 1} \frac{(-1)^{2k}}{4k^2-1} = f(\pi) = 0$$

$$\Rightarrow \sum_{k\geq 1} \frac{1}{4k^2-1} = -\frac{2}{\pi} \times \left(-\frac{\pi}{4}\right) = \frac{1}{2}$$

(D)

FAUX

$$\frac{1}{\pi}\int_0^{\pi} \cos^2 t \, dt = \frac{1}{\pi}\int_0^{\pi} \frac{1+\cos(2t)}{2} \, dt$$

$$= \frac{1}{2\pi}\int_0^{\pi} (1+\cos(2t)) \, dt$$

$$= \frac{1}{2\pi}\left[t + \frac{\sin(2t)}{2}\right]_0^{\pi}$$

$$\frac{1}{\pi}\int_0^{\pi} \cos^2 t \, dt = \frac{1}{2\pi}(\pi) = \frac{1}{2}$$

(E) Parseval :

VRAI

$$a_0^2 + \sum_{k\geq 1} \frac{a_k^2}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(t) \, dt$$

$$\frac{4}{\pi^2} + \frac{16}{2\pi^2} \sum_{k\geq 0} \frac{1}{(4k^2-1)^2} = \frac{1}{\pi}\int_0^{\pi} \cos^2(t/2) \, dt$$

$$= \frac{1}{2\pi}\int_0^{\pi} (1+\cos(t)) \, dt$$

$$= \frac{1}{2\pi}\left[t + \sin t\right]_0^{\pi}$$

$$\frac{4}{\pi^2} + \frac{8}{\pi^2} \sum_{k\geq 0} \frac{1}{(4k^2-1)^2} = \frac{1}{2} \iff \sum_{k\geq 1} \frac{1}{(4k^2-1)^2} = \left(\frac{1}{2} - \frac{4}{\pi^2}\right) \times \frac{\pi^2}{8}$$

$$= \frac{\pi^2}{16} - \frac{1}{2}$$